4001. (a) Differentiate with respect to y, and set $\frac{dx}{dy} = 0$.
   (b) The curve is a positive quartic, with x in terms of y. You might want to start by sketching the reflection in $y = x$.

4002. Draw a force diagram for the chock. You don't, in fact, need a force diagram for the barrel. Assume that friction between chock and slope is limiting, and resolve parallel and perpendicular to the slope.

4003. Use the factor theorem to produce two simpler DEs. Solve the first by direct integration, the second by writing down a standard general solution (or by separation of variables if needs be).

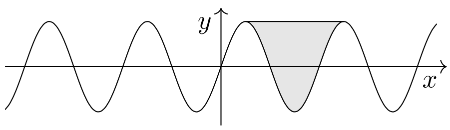
4004. Since the result to be proved is an implication in two directions, two proofs are required.
   ① In one, start with k triangular, so you can use the formula $T_n = \frac{1}{2}n(n+1)$. Simplify the algebra and prove that $8k + 1$ is a square.
   ② In the other, start with $8k+1$ being square. It must be the square of an odd number. Write this algebraically, prove that k is triangular.

4005. Use the double-angle identity
$$\cos 2x \equiv 2\cos^2 x - 1.$$
Consider a transformed image of $y = \cos x$.

4006. Differentiate to show that $\dfrac{dy}{dz} = \dfrac{1 + y^2}{1 - 2yz}$.

4007. Find the equation of the line in the standard form $y = mx + c$, and consider the value of c.

4008. Find the x coordinates of the first two maxima of $y = \sin 4x$ with $x \geq 0$.



Then use these as the limits of a definite integral, whose integrand is the y difference between the curve and the line.

4009. Use small-angle approximations, which can then be taken as exact in the limit as $x \to 0$.

4010. Sketch the graphs carefully. Use the symmetry to simplify your calculations. Find the x coordinate of the intersection in the positive quadrant, and set up a definite integral.

4011. There is one other value θ in the domain $[0, 2\pi)$ that the student has lost. This value is larger than those currently found.

4012. Solve the inequalities individually, and show that the intersection of the solution sets is the empty set. You might find a sketch of a number line or (x, y) axes useful.

4013. Solve for intersections, factorising the resulting equation.

4014. Without loss of generality, put two of the vertices at $(0, 0)$ and $(2, 0)$, and the other at (a, b), for $a, b > 0$. Show that the ratio of radii is minimised when $a = 1$. Hence, show that the ratio of radii is minimised when the triangle is equilateral.

4015. (a) Multiply three binomial probabilities. Note: $(2, 0, 1)$ represents an event, not an outcome. It refers to *any* game in which the numbers removed are 2 then 0 then 1.
   (b) The possibilities for numbers of dice removed on each turn are $(2, 0, 1)$, $(1, 1, 1)$, $(0, 2, 1)$, $(1, 0, 2)$, and two more.
   (c) Calculate the probability of each of the games from (b), and add.

4016. Solve numerically or with a polynomial solver. Then use the factor theorem. Note that one of the factors is an irreducible quadratic, and cannot be found using the (real) factor theorem. Find this by taking out the other factors.

4017. (a) The product is a positive sextic graph, with triple roots at $x = \pm 1$.
   (b) The sum is a positive cubic, with roots $x = \pm 1$. It must also, due to the symmetry between $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$, have a third root. Work out where this is, and sketch from the complete set of roots.
   (c) The cubics are both monic, so each has leading coefficient 1. In the difference $\mathrm{f}(x) - \mathrm{g}(x)$, the leading terms cancel, leaving a quadratic.

4018. Show that each group sums to at least $\frac{1}{2}$, and that there are infinitely many groups in the series.

4019. (a) Start with the RHS. Use two versions of the $\cos 2\theta$ double-angle formula, choosing them so as to eliminate the 1's on the top and bottom.
   (b) Multiply top and bottom of the integrand by $\cos 2\theta$. Then, write the resulting fraction in terms of the RHS of the identity from (a).

4020. (a) Include four forces.
   (b) Resolve parallel to the slope, using the fact that the acceleration is zero.

(c) With tension gone, resolve parallel to the slope again, and find the (negative) acceleration of the sledge. Then use a *suvat*.

(d) Reverse the direction of the friction, so that it acts up the slope. Calculate and interpret the new acceleration.

4021. (a) Solve simultaneously for $\text{f}(x)$ and $\text{g}(x)$.

(b) Simplify the LHS of the identity, using part (a). Then use the first Pythagorean trig identity.

4022. Find an expression for $\text{f}'(x)$. Then sub $y = A\,\text{f}(x)$ into the LHS. Factorise. Show that, if $y = A\,\text{f}(x)$ is to satisfy the DE, then $A^2 - A = 0$. Draw a conclusion from this.

4023. Rearrange to make y^2 (but not y) the subject. You can simplify things significantly this way: to show that y is stationary, it is enough to show that y^2 is stationary. So, set $\frac{d}{dx}(y^2) = 0$ and solve.

4024. In setting up for all three results, express the APs as $a_n = a + (n-1)c$ and $b_n = b + (n-1)d$.

(a) Show that the following difference is constant:

$$pa_{n+1} + qb_{n+1} - (pa_n + qb_n).$$

(b) Show that the following ratio is constant:

$$\frac{pa_{n+1} + qb_{n+1}}{pa_n + qb_n}.$$

(c) Divide top and bottom by n, before taking the limit $n \to \infty$.

4025. Draw a clear diagram to scale. Find the resultant of the two given forces; the third force must be the negative of this. Then, for rotational equilibrium, the three lines of action must be concurrent.

4026. Find the first derivative by the chain rule, making sure to simplify fully. You want a fraction, with no negative indices. Find the second derivative by the quotient rule. To simplify, divide top and bottom by $(r^2 - x^2)^{\frac{1}{2}}$.

4027. We assume a simple random sample. The set of data is large, so we assume that any datum has a 25% chance of lying in any particular quartile. This allows the use of a binomial distribution in both (a) and (b).

4028. Every point on the graph you draw must be a point on the graph given. But the converse is not true. Consider the inequality $x^4 - 5x^2 + 4 < 0$.

4029. The statement is true. Consider the fact that the graph $y = \text{f}(x)$ has an x intercept at $x = a$, which is also a stationary point.

4030. Using the second Pythagorean trig identity, write the equation as a quadratic in $\sec\theta$. Solve this for $\theta \in [0, 2\pi)$.

4031. (a) Split the forces applied to the ends of the strut into horizontal and vertical components. For the wheel, consider the NIII pair of the force acting downwards of the top of the strut. There are two struts.

(b) Consider vertical equilibrium for the wheel, to find the reaction acting vertically downwards on the right-hand strut. Then take moments around the base of the strut. For the contact force, you then need the Pythagorean sum of the two contact forces at at the base.

(c) Consider the fact that a spoke/string/cable can only pull, not push. Compression would cause buckling.

4032. Set up a limit of the gradient. Cancel a factor of $k - 1$ in the numerator and denominator before taking the limit.

4033. This is a quadratic in $x^{\frac{3}{2}}$.

4034. (a) This is $\mathbb{P}(X_1 = 1, 3)$. By symmetry, the two individual probabilities are equal.

(b) Calculate all probabilities in $X \sim B(4, 0.5)$, square them, and add them.

4035. Rewrite the integrand as $\dfrac{\frac{1}{x}}{\ln x}$.

4036. Rearrange each equation for $\cos^2 y$. You'll need the first Pythagorean trig identity in the first. Then subtract the equations and solve for $\cos x$. Once you've found x, substitute in to find y. You'll need to substitute into both equations to work out which values satisfy both.

4037. Use a combinatorial approach. The possibility space consists of the ${}^{12}\text{C}_6$ sets of edges which can be chosen. The task is to work out how many are successful. This is easiest if you spot that each such hexagon is defined by its normal, which is a space diagonal of the cube.

4038. (a) Differentiate implicitly and set $\frac{dy}{dx}$ to -1.

(b) Rearrange to make y the subject, and show that, as $x = \ln 2$, a ln function takes inputs tending to zero. Use symmetry to show that $y = \ln 2$ is also an asymptote.

(c) Show that $(0, 0)$ is the only axis intercept. Use this, (a) and (b), and the fact that the curve is symmetrical in the line $y = x$.

4039. Rewrite as a proper algebraic fraction. Then use the standard small-angle approximation, and the generalised binomial expansion.

4040. (a) This is true.
  (b) False. When looking for a counterexample, note that the result would hold if f and g were polynomial functions.

4041. Count up the numbers of outcomes (successful and total), classifying by the value of Z. This starts with $Z = 1$, which gives 16 outcomes, of which none are successful.

4042. Use the second Pythagorean trig identity.

4043. Use the factor theorem to write

$$\frac{dy}{dx} = k(x-2)(x-3)(x-4).$$

Expand this (keeping the k out) and integrate. Use the given values to find the constant of integration and k.

4044. (a) Solve for intersections.
  (b) Differentiate each with respect to t to find the velocities. Evaluate these at the t value found in (a), and verify that they are the same.

4045. Sketch the graph $y = |\sin\theta|$, and confirm that it has period π. Then integrate it on $[0, \pi]$, where you can deal with the mod sign easily. Divide the integral by the length of the domain.

4046. Use the factor theorem to factorise $\text{g}(x)$. Then equate coefficients on x^2 (you don't need to do the full expansion). This brings the sum of a GP.

4047. (a) Show that there are no axis intercepts. Find two SPs, by setting the derivative to zero. There is one vertical asymptote in the domain (and two vertical asymptotes which border the domain).
  (b) Draw the asymptotes first. Consider that, since $\operatorname{cosec}^2 x$ is always positive, in the vicinity of the asymptotes y tends to positive infinity.

4048. Solve the equation $3003x^3 + 1896x^2 + 51x - 90 = 0$ using a calculator or numerical method, then use the factor theorem. You will also need a constant factor.

4049. Use the substitution $u = x^4 - 1$. Then write $x^7\,dx$ as $x^4 \cdot x^3\,dx$ to facilitate the substitution.

4050. The points of tangency dictate the multiplicity of the roots of $\text{f}(x) - \text{g}(x) = 0$. Keep in mind that a quartic can have a maximum of four roots.

4051. Substitute the compound-angle formulae in. Then divide top and bottom of the fraction by $\sin\alpha\sin\beta$, and simplify.

4052. Rearrange to make $\sqrt{x}$ the subject, then square both sides. Sketch this graph first, then remove any sections of it which don't satisfy the original equation.

4053. Solve the inequality $10X - X^2 > 16$. Then, with $\{X \in \mathbb{R} : 0 \le X \le 10\}$ as the possibility space, considers the lengths of the relevant intervals.

4054. The mean $\bar{x}$ is $\frac{1}{2}(x_1 + x_2)$. So, find the sum of the two values (note the $\mp$ sign in the denominator) and then divide by two.

4055. Don't use an algebraic approach here. Find the prime factorisation of 120, and then seek out the relevant factors in the product.

4056. (a) Substitute $t = 0$.
  (b) Substitute $t = \frac{26}{3}$ into the first two terms (the linear part) of $\text{P}(t)$.
  (c) Set the derivative to zero for SPs. Use the graph to identify the particular local maxima required.
  (d) Set up an equation for the population reaching 6000. It is not analytically solvable. So, use the Newton-Raphson method.

4057. Put $1 - \frac{1}{i^2}$ over a common denominator. Factorise the numerator. Then write the product longhand, and show that almost all of the factors cancel.

4058. Use the factor theorem to establish that each of the equations is satisfied by the (x, y) points on two parallel lines. Find the four intersections of these pairs of parallel lines.

4059. Consider the centre $x = 2\cos\theta$, $y = 2\sin\theta$ as a parametric curve in its own right. This is a circle. The region you are looking for is an annulus.

4060. Differentiate implicitly wrt x. Set the derivative $\frac{dy}{dx} = 0$ without making it the subject. Sub the resulting linear equation into the equation of the curve.

4061. Place the person a distance d to the right of the right-hand support. Assume the ladder is on the point of tipping. This lets you set the reaction at the left-hand support to zero. Take moments to find d.

— Alternative Method —

Place the person at the end of the beam, calling the reactions R_1 and R_2. Solve to find R_1 and R_2, and interpret the negative value that arises.

4062. (a) Differentiate by the chain rule. Then use the second Pythagorean identity on the numerator of the fraction, and split the fraction up.

(b) Expand $\tan(x + \frac{\pi}{4})$ using a compound-angle formula, simplifying fully. Then reciprocate the result to get an expression for $\cot(x + \frac{\pi}{4})$. Add (common denominator) and simplify.

(c) Start with $2 \sec 2x$, and expand using the double-angle formula $\cos 2x \equiv \cos^2 x - \sin^2 x$. Then divide top and bottom of the fraction by $\cos^2 x$. Finally, use the second Pythagorean trig identity again.

4063. (a) Solve for intersections. Then find the gradients of the tangents to each curve at these points, and compare.

(b) This needs a reasonably accurate sketch. Find the coordinates of the points of intersection, and use the gradients

4064. Set up the following limit and definite integral:

$$\lim_{k\to\infty} \int_{-k}^{k} \mathrm{f}(x)\, dx.$$

4065. The product S_1S_2 is zero if either or both of S_1 and S_2 is zero. So, rewrite as

$$\mathbb{P}(S_1 = 1 \text{ and } S_2 = 1) = ...$$

Then convert into algebra and solve.

4066. (a) There's quite a bit of product rule here. Find the derivatives, simplifying fully as you go. Then substitute into the LHS of the DE.

(b) Consider the growth of the exponential term.

4067. Multiply up by the denominators, simplifying with a Pythagorean identity. Combine the sinusoids, writing in harmonic form. Solve the resulting equation, being careful to check whether any roots satisfy the original equations.

4068. Differentiate by the quotient rule to find x_1, x_2. You should get a surd expression. Substitute this expression back into f and simplify, rationalising the denominators of the surds that emerge.

4069. The implication is backwards. Prove this direction by assuming the integral statement and defining a function G such that $\mathrm{G}'(x) = \mathrm{g}(x)$. Disprove the other direction with a counterexample: consider the constant of integration.

4070. Draw a careful sketch. You don't need, in fact, to do any calculus. Consider rotational symmetry.

4071. Using a combinatorics method, the possibility space contains ${}^{52}\mathrm{C}_4$ possible hands. For successful hands, the suits are AABC. Work out the number of ways of choosing A, B and C. Having chosen e.g. spades, spades, clubs, hearts, work out how many ways there are of choosing the cards.

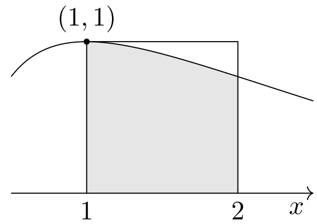
— Alternative Method —

Using a conditioning method, write P for Paired and S for Solo. Find the probability of suits PPSS in that order. Then multiply by the number of orders of PPSS.

4072. That fact the total energy $T + V$ is "conserved" means that it is constant, i.e. independent of t.

Use constant acceleration formulae to find T and V in terms of u, g, t. Add these, and show that t drops out of the resulting expression.

4073. Show that the scenario is as follows:



4074. For each, consider boundary cases:

(a) $X \subset Y \subset Z$ and $X \cap Y = \varnothing$,

(b) $X \subset Y \subset Z$ and $Y \cap Z = \varnothing$.

4075. (a) i. Find the coordinates of the vertex.

ii. The upper boundary of the range cannot exceed 1.

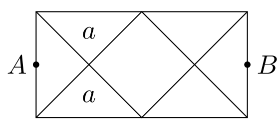
(b) i. Solve $P = \lambda P(1 - P)$.

ii. Show that, with $\lambda < 1$, the non-zero fixed point is outside $[0, 1]$.

iii. Show that the non-zero fixed point is at a maximum when λ is at a maximum. Use the value from (a) ii.

4076. Since x is small, high powers of x can be neglected. Expand binomially up to the term in x^2. Then substitute $x = \frac{1}{32}$.

4077. There are three regions which must be shaded. Consider the possibilities for the pair of regions marked a, and another like pair.



4078. (a) Consider the cross-section as a disc in the (x, y) plane. The axis of symmetry is a line parallel to the z axis.
(b) Find the (x, y) cross-sectional area A, and the z length l, and use $V = Al$.

4079. Let $\theta = \frac{1}{2}x$. Then start with the RHS, and use two versions of the $\cos 2\theta$ double-angle formula. In each case, you want to get rid of the 1.

4080. You are looking for symmetry in $y = \text{f}(x)$. If $x = k$ is a line of symmetry, then

$$\text{f}(k - x) \equiv \text{f}(k + x).$$

You can't find this line using roots, because there aren't any. Instead, use the stationary points. Identify k, then write $\text{f}(x)$ as a polynomial in $(x - k)$. This should make the symmetry explicit.

4081. Find vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$ in terms of q. You are told that these are parallel. So, equate gradients to bypass k and solve for q directly.

4082. Adding two of the equations will allow you to find one of the variables immediately.

4083. Answer the same question with the curve $x^2y^2 = 1$ first. You don't need to use calculus, or solve any equations, to find the circle. Then transform the problem, i.e. replace x by $2x - 1$ and y by $2y - 1$ in both question and answer.

4084. (a) Transform the components one by one: each T_i leaves one component unchanged.
(b) Attempt to solve $\mathbf{b} = \mathbf{a}$.
(c) i. Consider the effect on $(1, 0)$ and $(0, 1)$.
ii. Use the result of (b).

4085. The exact positions of the points aren't important. Only their order around the circumference is. So, consider the possibility space as the $4! = 24$ orders of $ABCD$.

4086. Write $u_0 = k$ to make the algebra easier. Find u_1, u_2, u_3 in terms of k. As you go, put each fraction over a common denominator and simplify fully, so that the algebra doesn't get out of hand.

4087. Consider generic stretches by scale factor p in the x direction and by q in the y direction, applied to $y = x^4 - x^2$:

$$y = q\big((x/p)^4 - (x/p)^2\big).$$

Expand this, equate coefficients, and solve to find p and q in terms of a.

4088. (a) Set up horizontal and vertical *suvat*s, for a pair of simultaneous equations in t and θ. From these eliminate t by subbing the horizontal into the vertical. Then use the second Pythagorean trig identity to convert $\sec^2\theta$ into $\tan^2\theta$.
(b) Solve the quadratic in $\tan\theta$.

4089. You need to show that $a^2 + b^2$ and $a^2 + c^2$ are both perfect squares. That $b^2 + c^2$ is a perfect square then follows by symmetry.

- For $a^2 + b^2$, expand the brackets and simplify. Use the fact that $p^2 + q^2 = r^2$ to cancel some terms, leaving only a perfect square.
- For $a^2 + c^2$, expand the brackets, simplify, and then factorise directly. You don't need to use $p^2 + q^2 = r^2$ here.

4090. Differentiate by the chain rule. Cancel factors of $\cos\frac{x}{2}$ in the denominator, and use the double-angle formula $\sin 2\theta \equiv 2\sin\theta\cos\theta$.

4091. Assume, for a contradiction, that prime numbers $p_1 < p_2 < p_3$ form a GP. Equate ratios and find a contradiction based on prime factorisation.

4092. Once a random number n is chosen, the number of X_i variables which then change is distributed binomially as $V \sim B(n, 1/2)$. Calculate $\mathbb{P}(V = 2)$ for $n = 2, 3, 4$.

4093. As $x \to \pm\infty$, the right-hand term tends to zero. There is also a double asymptote at $x = 0$.

4094. The number of squares threatened by the first knight depends on its proximity to the side of the board. In the corner, two squares are threatened; in the centre, eight are. So, use a conditioning approach, by location of the first knight.

4095. The first fact tells you that the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ form a closed quadrilateral when placed tip-to-tail. Establish that $\mathbf{b}$ and $\mathbf{d}$ must be opposite each other. Then resolve in the direction of $\mathbf{b}$. There are two answers.

4096. Differentiate by the quotient rule, and set the derivative to zero. Rearrange the equation to the form $2\tan x = ...$, and consider the behaviour of $y = \text{LHS}$ and $y = \text{RHS}$.

— Alternative Method —

Show that the curve has no discontinuities, and that it has infinitely many roots. Use the roots to construct the (approximate) location of infinitely many stationary points.

4097. Since f is quartic, $y = \mathrm{f}(x)$ must have at least one turning point. And a polynomial function f cannot be invertible over an interval containing a turning point. Put these facts together to show that $x = \alpha$ is a repeated root.

4098. Multiply by e^x, and you have a quadratic in e^{2x}.

4099. This is a separable DE. Rewrite using $y = \mathrm{f}(x)$, then separate variables and integrate. Rearrange your result (which will contain x and y and $+c$ or similar) into the required form. Since your $\mathrm{f}(x)$ is a solution curve, you can then put any x value you like into it.

4100. Take the ten letters as distinct entities. Quote a result for the number of entries in a list of possible rearrangements. Then consider the overcounting factor: the number of times that any particular arrangement, e.g. LOVELINESS itself, appears in the list.

——— End of 41st Hundred ———